AD-A096 889 WISCONSIN UNIV MADISON F/G 12/1
IMPROVED SUDAKOV-TYPE BOUNDS FOR OPTIMAL CONFIDENCE LIMITS ON T--ETC(U)
FEB 81 B HARRIS, A P SOMS N00014-79-C-0321
UNCLASSIFIED TR-637 NL

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(12)

DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN

Madison, Wisconsin 53706

TECHNICAL REPORT NO. 637

February 1981

IMPROVED SUDAKOV-TYPE BOUNDS FOR OPTIMAL CONFIDENCE LIMITS ON THE RELIABILITY OF SERIES SYSTEMS

by

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Improved Sudakov-Type Bounds for
Optimal Confidence Limits on the Reliability
of Series Systems

Bernard Harris[*] and Andrew P. Soms[**]

Abstract

A sharper Sudakov-type lower bound for the lower confidence limit on the reliability of a series system than the one given in Harris and Soms (1980) is obtained. Numerical examples, coverage probabilities and the listings of the short FORTRAN programs used are also provided.

Key words: Lindstrom-Madden approximation; Optimal confidence bounds; reliability; Series system.

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Research supported by the Office of Naval Research under Contract No. N00014-79-C-0321 and the United States Army under Contract No. DAAG29-80-C-0041.

1. The Improved Sudakov-Type Bound

We will adhere to the notation of Harris and Soms (1980) and we refer the reader to this report for background material and additional references. Let Y_i, $1 \le i \le k$, be independent binomial random variables with parameters n_i and p_i (the success probability) and let $X_i = n_i - Y_i$, $q_i = 1 - p_i$, $1 \le i \le k$, with $n_1 \le n_2 \le \ldots \le n_k$ here and throughout. It is desired to obtain a "good" lower confidence limit on $\prod_{i=1}^{k} p_i$, the reliability of a series system with independent components. Let $g(\tilde{x}) = g(x_1, x_2, \ldots, x_k)$ be a monotonically decreasing or increasing (in each component) ordering function for real (not necessarily integral) x_i, $0 \le x_i \le n_i$, with large or small values, respectively, of $g(\tilde{x})$ being best. Let $r_1 > r_2 > \ldots > r_s$ be the ordered values of $g(\tilde{x})$ in the decreasing case and $r_1 < r_2 < \ldots < r_s$ in the increasing and let $A_i = \left\{\tilde{x} \mid g(\tilde{x}) = r_i\right\}$, $i = 1,2,\ldots,s$. Then $(A_1, A_2, \ldots, A_s)$ is a monotonic partition, i.e., $(0,0,\ldots,0) \in A_1$, $(n_1, n_2, \ldots, n_k) \in A_s$ and if $\tilde{x}_1 = (x_{11}, \ldots, x_{1k})$, $\tilde{x}_2 = (x_{21}, \ldots, x_{2k})$ with $x_{1i} \le x_{2i}$, $i = 1,2,\ldots,k$, then $\tilde{x}_1 \in A_i$ implies $\tilde{x}_2 \in A_j$, $j \ge i$.

Let

$$f(\tilde{x};\tilde{p}) = P_{\tilde{p}}(\tilde{X} = \tilde{x}) = \prod_{i=1}^{k} \binom{n_i}{x_i} p_i^{n_i - x_i} q_i^{x_i} = \prod_{i=1}^{k} \binom{n_i}{y_i} p_i^{y_i} q_i^{n_i - y_i} \quad (2.1)$$

and for $1 \le n \le s-1$, let

$$a_n = \inf\left\{ \prod_{i=1}^{k} p_i \,\middle|\, \sum_{x_i \in A_i, i \le n} f(\tilde{x}_i;\tilde{p}) = \alpha \right\} \quad (2.2)$$

and $a_s = 0$. Each such partition may be identified with a function defined on the set of sample outcomes by defining the ordering

function $g(\tilde{x})$, where

$$g(\tilde{x}) = n \quad \text{if} \quad \tilde{x} \in A_n , \qquad 1 \leq n \leq s ; \tag{2.3}$$

obviously $g(\tilde{x})$ inherits the monotonicity properties of the partition.

For defining Buehler's (1957) method of optimal lower confidence intervals on $\prod_{i=1}^{k} p_i$ we assume that $g(\tilde{x})$ has been redefined, if necessary, as in (2.3). Then we have, from Harris and Soms (1980),

<u>Theorem 1</u>. Let $\tilde{X}$ be distributed by (2.1). Then $a_{g(\tilde{X})}$ is a $(1-\alpha)$ lower confidence bound for $\prod_{i=1}^{k} p_i$. If $b_{g(\tilde{X})}$ is also a $(1-\alpha)$ lower confidence bound for $\prod_{i=1}^{k} p_i$ which is monotonically increasing in $g(\tilde{x})$, then $b_i \leq a_i$, $1 \leq i \leq s$.

We now let $g(\tilde{x})$ denote the original ordering function, since this is necessary for the applications below. In order to obtain bounds for $a_{g(\tilde{x}_0)}$ we must assume that $\tilde{x}_0$ is such that for each $t = 1,2,\ldots,k$, the equation

$$g(i_1, i_2, \ldots, i_{t-1}, y_t, 0, \ldots, 0) = g(\tilde{x}_0) \tag{2.4}$$

has a unique solution y_t, $y_t < n_t$, where i_r, $r = 1,2,\ldots,t-1$, are integers, $0 \leq i_r \leq y_r$. Define y_i^* by $g(0,0,\ldots,0,y_i^*,0,\ldots,0) = g(\tilde{x}_0)$, where y_i^* is in the i^{th} position, the rest of the arguments being 0. Note that $y_1 = y_1^*$. A sufficient condition for (2.4) to hold is that

$$\left.\begin{array}{l} g(\tilde{x}) \text{ is strictly monotonic for } 0 \leq x_j \leq y_j^* < n_j,\ j=1,2,\ldots,k \\ \text{and} \\ \qquad \lim\limits_{x_r \to n_r} g(\tilde{x}) < g(\tilde{x}_0) \text{ for } g(\tilde{x}) \text{ decreasing} \\ \text{or} \\ \qquad \lim\limits_{x_r \to n_r} g(\tilde{x}) > g(\tilde{x}_0) \text{ for } g(\tilde{x}) \text{ increasing, } r=1,2,\ldots,k . \end{array}\right\} \tag{2.5}$$

Assume now that (2.5) is satisfied and that in addition

$$\frac{y_r - i_r}{n_r - i_r} \geq \frac{y_{r+1}}{n_{r+1}} , \qquad r=1,2,\ldots,k-1 , \tag{2.6}$$

where i_r integral, $0 \leq i_r \leq y_r$. Let

$$I_p(r,s) = \frac{1}{B(r,s)} \int_0^p t^{r-1}(1-t)^{s-1} dt ,$$

and for $0 \leq y < n$, real, define $u(n,y,\alpha)$ by $\alpha = I_{u(n,y,\alpha)}(n-y, y+1)$. Then it was shown in Harris and Soms (1980) that

$$u(n_1, y_1, \alpha) \leq a_{g(\tilde{x}_0)} \leq \min_{1 \leq i \leq k} u(n_i, [y_i^*], \alpha) , \tag{2.7}$$

and thus if y_1 is an integer, $a_{g(\tilde{x}_0)} = u(n_1, y_1, \alpha)$.

Thus a conservative procedure is to use $u(n_1, y_1, \alpha)$ for $a_{g(\tilde{x}_0)}$. It is generally believed that this is quite conservative (see, e.g., Mann, Schafer and Singpurwalla, 1974). It is thus of practical interest to inquire whether the lower bound in (2.7) can be tightened. An examination of the proof of (2.7) in Harris and Soms (1980) shows this to be the case, with the new proof being coincident with the previous one, except for the omission of the final step. We have

__Theorem 2__. Under the same assumptions as for (2.7),

$$u'(n_1,n_2,y_1,y_2,\alpha) \le a_{g(\tilde{x}_0)} \le \min_{1\le i\le k} u(n_1,[y_i^*],\alpha) ,$$

where $u'(n_1,n_2,y_1,y_2,\alpha)$ is the solution in a of the equation

$$\sup_{\prod_{i=1}^{k} p_i = a} \sum_{i_1=0}^{[y_1]} \binom{n_1}{n_1-i_1} p_1^{n_1-i_1} q_1^{i_1} I_{\prod_{i=2}^{k} p_i}(n_2-y_2,y_2+1) = \alpha .$$

It follows from the proof in Harris and Soms (1980) that

$$u(n_1,y_1,\alpha) \le u'(n_1,n_2,y_1,y_2,\alpha) .$$

Thus the only question is how great the improvement will be in using u'. The numerical examples in 2. show that this can be substantial.

__Remarks__. u' can be calculated quickly and efficiently by using a short FORTRAN program. The listing is given in the Appendix along with the listing of the program used to calculate coverage probabilities. Two ordering functions that satisfy (2.5) and (2.6) are $g(\tilde{x}) = \prod_{i=1}^{k}((n_i-x_i)/n_i)$ if $g(\tilde{x}_0)>0$ and $g(\tilde{x}) = \sum_{i=1}^{k} x_i/n_i$ for sufficiently large n_i, $1 \le i \le k$ (see Harris and Soms, 1980, for details).

2. Coverage Probabilities and Numerical Examples

The ordering function used here throughout is $g(\tilde{x}) = \prod_{i=1}^{k}((n_i-x_i)/n_i)$. Table 1 gives the coverage probabilities for $k=3$, $\tilde{n}=(5,7,10)$, $\alpha = .10$ and selected $\tilde{p} = (p_1,p_2,p_3)$, for both the lower LB and upper UB bounds of (2.7). While the optimality property of Theorem 1 implies that there are $\tilde{p}$ for which the coverage probability is less than .9, it does not seem that there are very many such $\tilde{p}$.

[Insert Table 1 here.]

Table 2 gives some comparisons of LB, UB, the improved lower bound LBI and the true value TV for $k=2$, $\alpha = .05$ and selected $\tilde{n} = (n_1,n_2)$ and $\tilde{x}_0 = (x_{01},x_{02})$. LBI gives substantial improvement when n_1 is small compared to n_2 and little or none if n_1 is approximately the same as n_2. The TV for $\tilde{n} = (5,5)$ and $\tilde{x}_0 = (1,1)$ agrees with the value in Lipow and Riley (1959).

[Insert Table 2 here.]

Table 3 gives LB, UB and LBI for $k=5$, $\alpha = .05$ and selected $\tilde{n}$ and $\tilde{x}$.

[Insert Table 3 here.]

References

Buehler, Robert J. (1957), "Confidence Limits for the Product of Two Binomial Parameters", Journal of the American Statistical Association, 52, 482-93.

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Mann, Nancy R., Schafer, Ray E., and Singpurwalla, Nozer D. (1974), Methods for Statistical Analysis of Reliability and Life Data, New York: John Wiley and Sons.

The listings of the coverage and improved bound FORTRAN programs are given below.

```
C SERIES WITH UNLIKE COMPONENTS
C MDBIN, MDBETA, MDBETI IMSL BINOMIAL, BETA, INVERSE BETA ROUTINES,
C N'S SAMPLE SIZES FROM SMALLEST TO LARGEST, X'S FAILURES, NINT NUMBER
C OF P VALUES CONSIDERED FOR SUP, EPS MAXIMUM ERROR IN SOLUTION FOR
C REFINED BOUND, NO X'S EQUAL TO N'S
      DIMENSION N(50),X(50)
  100 FORMAT ( )
      DATA EPS/.001/
    1 READ 100,K,ALPHA,NINT
      IF (K.EQ.0) GO TO 99
      READ 100,(N(I),X(I),I=1,K)
      ITEM1=1
      ITEM2=1
      DO 2 I=1,K
      ITEM1=ITEM1*N(I)
    2 ITEM2=ITEM2*(N(I)-X(I))
      ITEM3=N(1)*(ITEM1-ITEM2)
      Y1=ITEM3/FLOAT(ITEM1)
      NY1=ITEM3/ITEM1
      CALL MDBETI(ALPHA,N(1)-Y1,Y1+1.,A,IER)
      A1=A
C COMPUTATION OF UPPER BOUND
      BU=1.
      DO 3 I=1,K
      ITEMT=N(I)*(ITEM1-ITEM2)
      NYT=ITEMT/ITEM1
      CALL MDBETI(ALPHA,FLOAT(N(I)-NYT),NYT+1.,BUT,IER)
    3 IF (BUT.LT.BU) BU=BUT
      IF (Y1-NY1.LE..01) GO TO 52
C COMPUTATION OF IMPROVED LOWER BOUND
      TEML=A
      TEMU=BU
    9 TEM=(TEML+TEMU)/2.
      DELTA=(1-TEM)/NINT
      SUP=0.
      DO 82 I=1,NINT
      P1=TEM+(I-1)*DELTA
      P2=TEM/P1
      Q1=1-P1
      PROB=0.
      DO 33 K1=0,NY1
      Y2=N(2)-(ITEM2/FLOAT(ITEM1))*N(1)*N(2)/(FLOAT(N(1)-K1))
      CALL MDBIN(K1,N(1),Q1,PS1,PK1,IER)
      CALL MDBETA(P2,N(2)-Y2,Y2+1.,PP2,IER)
   33 PROB=PROB+PK1*PP2
      IF (PROB.GT.SUP) SUP=PROB
   82 CONTINUE
      IF (SUP.LE.ALPHA) TEML=TEM
      IF (SUP.GT.ALPHA) TEMU=TEM
      IF (ABS(TEMU-TEML).LE. EPS) GO TO 10
      GO TO 9
   10 A=TEML
   52 PRINT 100,(N(I),X(I),I=1,K)
C LINDSTROM-MADDEN
      PRINT 100,ALPHA,A1,BU
C IMPROVED LINDSTROM-MADDEN LOWER BOUND, IF SAME, Y1 IS CLOSE TO INTEGER
      PRINT 100,A
      GO TO 1
   99 STOP
      END
```

```
C COVERAGE PROBABILITIES
C N'S SAMPLE SIZES FROM SMALLEST TO LARGEST, P'S SUCCESS PROBABILITIES,
C K NUMBER OF SUBSYSTEMS,1-ALPHA CONFIDENCE LEVEL
      DIMENSION P(10),N(20),NY(10),PLIMU(10),PI(10,81),LL(10),II(10)
  100 FORMAT ( )
      EQUIVALENCE (I1,II(1)),(I2,II(2)),(I3,II(3)),(I4,II(4)),
     1(I5,II(5)),(I6,II(6)),(I7,II(7)),(I8,II(8)),(I9,II(9)),(I10,II(10)
     2)
    1 READ 100,K,ALPHA
      IF (K.EQ.0) GO TO 21
      DO 221 I=1,10
      LL(I)=0
  221 N(I)=0
      READ 100,(N(I),P(I),I=1,K)
      DO 343 II1=1,K
      DO 343 II2=0,N(II1)
      CALL MDBIN(II2,N(II1),P(II1),PT,PI(II1,II2+1),IER)
  343 IF (PT.LE..0001) LL(II1)=LL(II1)+1
      PARAM=1.
      IPROD2=1
      DO 11 I=1,K
      IPROD2=IPROD2*N(I)
   11 PARAM=PARAM*P(I)
      PRL=0.
      PRU=0.
      DO 122 I1=LL(1),N(1)
      DO 122 I2=LL(2),N(2)
      DO 122 I3=LL(3),N(3)
      DO 122 I4=LL(4),N(4)
      DO 122 I5=LL(5),N(5)
      DO 122 I6=LL(6),N(6)
      DO 122 I7=LL(7),N(7)
      DO 122 I8=LL(8),N(8)
      DO 122 I9=LL(9),N(9)
      DO 122 I10=LL(10),N(10)
      PROBT=1.
      DO 511 IT=1,K
      INT=II(IT)
  511 PROBT=PROBT*PI(IT,INT+1)
      IPROD1=1
      DO 4 IJ=1,K
      IN=II(IJ)
    4 IPROD1=IPROD1*IN
      IF (IPROD1.EQ.0) GO TO 15
      ITEM=IPROD2-IPROD1
      Y1=N(1)*ITEM/FLOAT(IPROD2)
      DO 5 IT=1,K
      NY(IT)=N(IT)*ITEM/IPROD2
      A=FLOAT(N(IT)-NY(IT))
      B=FLOAT(NY(IT)+1)
    5 CALL MDBETA(PARAM,A,B,PLIMU(IT),IER)
      CALL MDBETA(PARAM,N(1)-Y1,Y1+1.,PLIML,IER)
      IF (PLIML.GE.ALPHA) PPL=PRL+PROBT
      IFLAG=0
      DO 6 IT=1,K
    6 IF (PLIMU(IT).GE.ALPHA) IFLAG=1
      IF (IFLAG.EQ.1.) PRU=PRU+PROBT
      GO TO 122
   15 PRL=PRL+PROBT
      PRU=PRU+PROBT
  122 CONTINUE
      PRINT 100,K,ALPHA,(N(I),P(I),I=1,K)
      PRINT 100,PRU,PPL
      GO TO 1
   21 STOP
      END
```

1. Coverage Probabilities for $\tilde{n}$ = (5,7,10) and α = .10

$\tilde{p}$	Coverages UB	Coverages LB
(.95,.95,.95)	.9999	.9999
(.95,.95,.90)	1.0000	1.0000
(.95,.95,.85)	1.0000	1.0000
(.95,.95,.80)	.9999	.9999
(.95,.95,.75)	.9999	.9999
(.95,.95,.70)	1.0000	1.0000
(.95,.95,.65)	.9535	.9927
(.95,.95,.60)	.9737	.9967
(.95,.95,.55)	.9869	.9874
(.95,.95,.50)	.9940	.9940

2. Comparison of Bounds and True Value for $k = 2$ and $\alpha = .05$

$\tilde{n}$	$\tilde{x}_o$	LB	LBI	TV	UB
(5,5)	(1,1)	.2166	.2166	.2776	.3426
(5,10)	(1,1)	.2761	.3317	.3317	.3426
(5,10)	(2,3)	.0859	.1521	.1529	.1893
(10,20)	(1,2)	.5037	.5675	.5691	.6058
(10,20)	(4,5)	.1851	.2137	.2137	.2224

3. Comparison of Bounds for $k = 5$ and $\alpha = .05$

$\tilde{n}$	$\tilde{x}_o$	LB	LBI	UB
(10,10,10,10,10)	(1,2,0,1,1)	.2893	.2893	.3035
(10,15,20,25,30)	(1,2,1,1,2)	.3599	.3871	.3934
(10,15,20,25,30)	(1,1,2,3,4)	.2838	.3017	.3035
(10,15,20,25,30)	(2,3,4,4,4)	.1319	.1478	.1500

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
TR No. 637

2. GOVT ACCESSION NO.
AD-A096

3. RECIPIENT'S CATALOG NUMBER
889

4. TITLE (and Subtitle)
6 Improved Sudakov-Type Bounds for Optimal Confidence Limits on the Reliability of Series Systems.

5. TYPE OF REPORT & PERIOD COVERED
Scientific-Interim

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
10 Bernard/Harris ~~and~~ Andrew P./Soms

8. CONTRACT OR GRANT NUMBER(s)
15 [redacted] N00014-79-C-0321,
[redacted] DAAG29-80-C-0041

12) 17

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Statistics & Mathematics Res. Ctr.
University of Wisconsin
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11) 25 Feb 81

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

14) TR-637

12. REPORT DATE
February 25, 1981

13. NUMBER OF PAGES
12 pages

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
9) Technical rept.

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this document is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lindstrom-Madden approximation; Optimal confidence bounds; Reliability; Series system.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified 380200 slt

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